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A Technique for Optimum Final Value Control of Powered Flight Trajectories (Revision No. 1)

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CALIFORNIA INSTITUTE OF TECHNOLOGY
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A Technique for Optimum Final Value Control of Powered Flight Trajectories (Revision No. 1)

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ABSTRACT

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A technique is described for steering a rocket vehicle to thrust termination where the objective is to minimize the sum of the squares of the variations of the standard burnout conditions. The analysis is based upon a geometrical interpretation of an optimal trajectory, which is constructed from a simplified mathematical model of the motion of the vehicle. It is shown that there is an envelope of reachable points in the space of boundary condition variations and that the control (steering) to be applied can be found by dropping a perpendicular to this envelope. An example of ascent guidance into near-Earth satellite orbit is worked out in detail, and numerical results are presented. It is shown that the control scheme is stable at the final time, and the relationship to the well known velocity-to-be-gained steering scheme is pointed out.

I. INTRODUCTION

Final value control of a rocket vehicle during powered flight consists of varying the thrust attitude (steering angle) program in such a way that, for a given set of perturbed position and velocity coordinates at a given initial time t_0, the mission objectives will be achieved at the final time T. The purpose of this Report is to describe a "least squares" technique for analytically determining the steering angle program. It is assumed that the vehicle's motion is restricted to a plane and that the time varying thrust acceleration level is given. We follow the well known approach of defining a state vector composed of the position and velocity coordinates of the vehicle and deal with variations of the state vector from some given preflight standard (nominal) trajectory. The analysis is carried out for the case of final time T being fixed, but the extension to the variable time case is outlined.

The least squares final value control technique is developed in Part V (Ref. 1). In Part VI an application to a specific rocket guidance problem is described and analyzed in detail. How this approach applies to the well known technique of velocity-to-be-gained steering is shown in Part VIII. The similar control schemes of Ref. 2 and 3 are numerically compared.

The notation employed is as follows: the independent variable is the time t; T is the fixed final time; other capital Roman letters are matrices; column vectors are indicated by a bar (–) over a small letter; the transcript of a vector or matrix is indicated by a superscript prime ($'$); and a subscript s refers to a time dependent quantity evaluated on the standard trajectory. A superscript 0 refers to a variation in the end conditions arising only from a state vector variation at the initial time t_0, assuming no control variation is applied between t_0 and T. The notation $\delta\bar{x}(t)$ indicates a variation from the standard value at the given time t; that is, $\delta\bar{x}(t) = \bar{x}(t) - \bar{x}_s(t)$. The notation (t) occasionally will be omitted from equations in order to simplify the exposition.

II. FORMULATION OF THE PROBLEM

Let the motion of the rocket vehicle be restricted to a plane, with equations of motion given by (Fig. 1)

$$\frac{d}{dt}(\bar{x}) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} a\cos\theta - \left(\dfrac{kx_3}{|\bar{r}|^3}\right) \\ a\sin\theta - \left(\dfrac{kx_4}{|\bar{r}|^3}\right) \\ x_1 \\ x_2 \end{bmatrix} \triangleq \bar{f}(\bar{x}, \theta, t) \qquad (1)$$

where

$[x_1, x_2] = \bar{v}' =$ the velocity vector of the vehicle; $[x_3, x_4] = \bar{r}' =$ the position vector of the vehicle ($\dot{\bar{r}} = \bar{v}$); $a(t)$ is the time varying prespecified thrust acceleration level; k is the gravitational constant; $\theta(t)$ is the steering angle to be determined when given the initial condition $\bar{x}(t_0)$, where $t \geq t_0$; $\bar{x}$ is the state vector. We imagine that there exists a preflight standard (nominal) trajectory which has been optimized in the sense that the standard value of the steering angle $\theta_s(t)$ has been chosen so that at the fixed final time T the functional

$$\beta_0 = \beta_0\,[x_1(T), x_2(T), x_3(T), x_4(T)] \qquad (2)$$

is a minimum, subject to

$$\beta_i = \beta_i\,[x_1(T), x_2(T), x_3(T), x_4(T)] = 0 \qquad (3)$$

$$i = 1, \ldots, r \leq 3$$

where the "boundary functions" β_i are some functions of the end conditions.[1] We assume that all β_i have the same physical dimensions in order to make intuitive sense out of the forthcoming analysis. Since this implies some arbitrary scaling of the β_i, we have thus chosen a metric for the "boundary function space" whose coordinates are β_i. The definition of the metric is arbitrary and can be shown to not affect the optimality of the standard trajectory. (The physical nature of the problem often suggests an appropriate selection.)

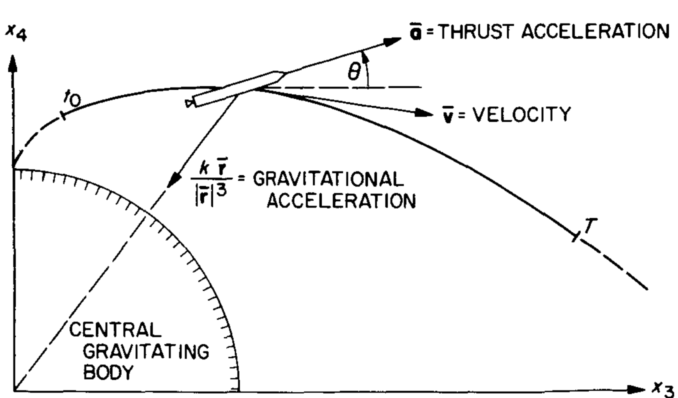

Fig. 1. The powered flight trajectory

[1]This is the Mayer formulation of the problem. Note that a problem of the type $\beta_i = \int_{t_0}^{T} \dot{\beta}_i\,[\bar{x}(t)]\,dt$ can also be formulated in this way by redefining the additional state variables $\dot{x}_{4+i} = \dot{\beta}_i$ for each such functional β_i. This generalization will not be treated here, however.

For the convenience of the subsequent analysis we define the control variable to be

$$y = \theta\,[\alpha(t)]^{-1} \tag{4}$$

where $\alpha(t)$ is a given time-varying normalizing function. The definition of $\alpha(t)$ is introduced below, where the motivation is apparent.

Let us consider variations from the standard trajectory, described by the perturbation equation (Ref. 4 and 5)

$$\frac{d}{dt}(\delta\bar{x}) = F\delta\bar{x} + G\delta y + \frac{1}{2}H\delta y^2 + \left[\begin{matrix}\text{terms of the order}\\ \text{of magnitude}\end{matrix}\ \left|\frac{k\delta x_i\,\delta x_j}{|\bar{r}|^4}\right|\right] + \text{higher order terms} \tag{5}$$

where

$$\delta\bar{x}(t) = \bar{x}(t) - \bar{x}(t)_{\text{standard}}$$

$$\delta y(t) = y(t) - y(t)_{\text{standard}}$$

$$F = \left[\frac{\partial\bar{f}}{\partial\bar{x}}\right] = \begin{bmatrix} 0 & 0 & 0 & \left(\frac{k}{|\bar{r}|^5}\right)(x_4^2 - 2x_3^2) & -\left(\frac{3kx_3x_4}{|\bar{r}|^5}\right) \\ 0 & 0 & 0 & -\left(\frac{3kx_3x_4}{|\bar{r}|^5}\right) & \left(\frac{k}{|\bar{r}|^5}\right)(x_3^2 - 2x_4^2) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

$$G = \left[\frac{\partial\bar{f}}{\partial y}\right] = \left(\frac{\partial\bar{f}}{\partial\theta}\right)\left(\frac{d\theta}{dy}\right) = (\alpha)\begin{bmatrix} -a\sin\theta \\ a\cos\theta \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$H = \left[\frac{\partial^2\bar{f}}{\partial y^2}\right] = \frac{\partial}{\partial\theta}\left(\frac{\partial G}{\partial y}\right)\left(\frac{d\theta}{dy}\right) = (\alpha^2)\begin{bmatrix} -a\cos\theta \\ -a\sin\theta \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

The matrices $F(t)$, $G(t)$, and $H(t)$ are evaluated along the standard trajectory by substituting into Eq. (6) through (8) the standard values $\bar{x}_s(t)$ and $\theta_s(t)$. We henceforth assume that the terms of order of magnitude $\left|\frac{k\,\delta x_i\,\delta x_j}{|\bar{r}|^4}\right|$ and the higher order terms are negligibly small compared to the other quantities on the right-hand side of Eq. (5); that is, the perturbed trajectory of the spacecraft remains sufficiently close to the standard path to allow the second order state variable deviations to be neglected. Thus we consider the modified second order perturbation equation consisting of the first three terms on the right-hand side of Eq. (5). We introduce the well known state transition matrix $U(T, t)$, defined by

$$\frac{d}{dt}U(T,t) = -\,U(T,t)\,F(t) \tag{9}$$

$$U(T,T) = \text{the identity}$$

from which it follows that

$$\delta\beta_i = \bar{\lambda}_i'(t_0)\,\delta\bar{x}(t_0) + \int_{t_0}^{T}\left[\eta_i\,\delta y + \xi_i\frac{\delta y^2}{2}\right]dt \qquad (10)$$

$$+ \text{ negligible terms} \qquad i = 0, 1, \ldots, r$$

where

$$\bar{\lambda}_i'(t) = \left[\left(\frac{\partial\beta_i}{\partial x_1(T)}\right) - - \left(\frac{\partial\beta_i}{\partial x_4(T)}\right)\right][U(T,t)]$$

$$\eta_i(t) = \bar{\lambda}_i'\,G(t)$$

$$\xi_i(t) = \bar{\lambda}_i'(t)\,H(t)$$

$\eta_i(t)$ is the impulse response function for the boundary function β_i, and $\xi_i(t)$ is the weighting function for the boundary function β_i. Equation (10) is the second order functional expansion which is exploited in this Report. The final value control problem consists of employing Eq. (10) to determine the $\delta y(t)$ for any given $\delta\bar{x}(t_0)$.

Note that we are considering only small deviations in the control variable in this Report; that is, $|\delta y(t)| = |y(t) - y(t)_s| \le \varepsilon$, so that the second order terms in the functional expansion of the penalty function dominate the higher order terms. By this restriction, we are dealing with what is called a "weak" variation of $y(t)$ in the terminology of the classical calculus of variations.

III. A GEOMETRICAL INTERPRETATION OF OPTIMALITY

In this Section we summarize the geometrical interpretation of optimality discussed in Ref. 1, 6, and 7.

Suppose we construct an $r + 1$ by $r + 1$ dimensional Euclidean "boundary function" space (Fig. 2) with coordinates $\delta\beta_i$ as given by Eq. (10). Thus, the $r + 1$ dimensional vector $\delta\bar{\beta}$ is given by

$$\delta\bar{\beta} = \delta\bar{\beta}^0 + \Delta\bar{\beta} \qquad (11)$$

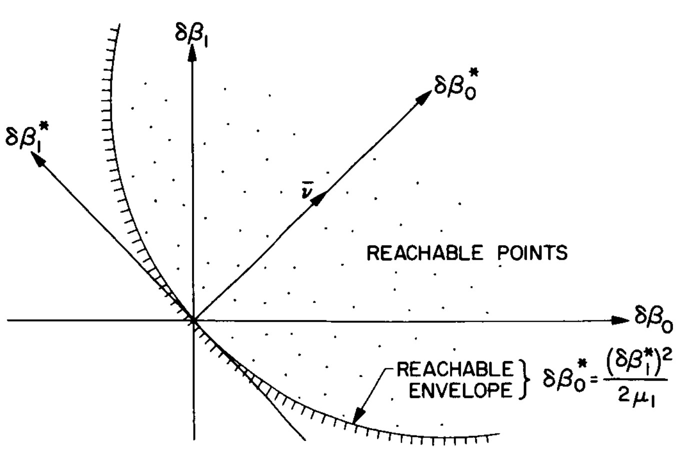

Fig. 2. The boundary function space for r = 1

where $\delta\bar{\beta}^0$ is defined to be the effect of the initial condition disturbance, that is,

$$\delta\bar{\beta}^0 = \left[\frac{\partial\bar{\beta}[\bar{x}(T)]}{\partial\bar{x}(t_0)}\right]\delta\bar{x}(t_0) \qquad (12)$$

and the $\Delta\bar{\beta}$ is defined to be the effect of varying the control over the interval (t_0, T), that is,

$$\Delta\bar{\beta}' = [\Delta\beta_0, \Delta\beta_1, \ldots, \Delta\beta_r] = \int_{t_0}^{T}\bar{\eta}'(t)\,\delta y(t)\,dt + \frac{1}{2}\int_{t_0}^{T}\bar{\xi}'(t)\,\delta y^2(t)\,dt \qquad (13)$$

$$+ \text{ negligible terms}$$

where

$$\bar{\eta}'(t) = [\eta_0(t), \eta_1(t), \ldots, \eta_r(t)]$$

$$\bar{\xi}'(t) = [\xi_0(t), \xi_1(t), \ldots, \xi_r(t)]$$

The standard trajectory corresponds to $\delta\bar{\beta} = 0$. The analysis of final value control from the geometrical point of view consists of finding those points in the boundary function space that can be reached by applying an arbitrary control $\delta y(t)$ over the interval (t_0, T) and determining the appropriate $\delta y(t)$ for a given $\delta\bar{\beta}^0$. Clearly all points in the neighborhood of $\delta\bar{\beta} = 0$ are not reachable if the standard trajectory has been designed so as to minimize β_0, subject to the end conditions; for this would imply that a point could be attained where $\delta\beta_0$ is a negative number and $\delta\beta_i = 0$ for $i = 1, 2, \ldots, r$, which is a contradiction. This intuitive notion leads directly to necessary conditions for optimality.

It is shown in Ref. 1 and 6 that, if the standard trajectory is optimal, there must exist for all times $0 \le t_0 \le T$ an orthonormal transformation (rotation) of the boundary function space which causes a linear combination of the influence functions to be zero. Thus, if

$$\delta\bar{\beta}^*_0 = L(t_0)\,\delta\bar{\beta} \tag{14}$$

where $L(t_0)$ is the $r+1$ by $r+1$ orthonormal matrix which diagonalizes the matrix

$$M = \int_{t_0}^{T} \bar{\eta}(t)\,\bar{\eta}'(t)\,dt \tag{15}$$

then we have

$$\delta\beta^*_0 = (\delta\beta^*_0)^0 + \frac{1}{2}\int_{t_0}^{T} \xi^*_0(t)\,\delta y^2(t)\,dt + \text{negligible terms} \tag{16}$$

$$\delta\beta^*_i = (\delta\beta^*_i)^0 + \int_{t_0}^{T} \eta^*_i(t)\,\delta y(t)\,dt + \text{negligible terms} \qquad i = 1, 2, \ldots, r \tag{17}$$

Thus, in the (*) coordinate system the impulse response function $\eta^*_0(t)$ is identically zero, that is,

$$\eta^*_0(t) = \sum_{i=0}^{r} \nu_i\,\eta_i(t) = 0 \tag{18}$$

where

$\{\nu_i\}$ are the elements of the first row of L. Indeed, Eq. (18) is the first necessary condition of the calculus of variations (equivalent to the Euler-Lagrange equations), and $\{\nu_i\}$ are the well known Lagrange multipliers.[2] Furthermore, it follows that[3]

$$\xi^*_0(t) = \sum_{i=0}^{r} \nu_i\,\xi_i(t) = \alpha^2(t)\sum_{i=0}^{r} \nu_i\bar{\lambda}'_i(t)\left[\frac{\partial^2 \bar{\mathrm{f}}}{\partial\theta^2}\right](t) \ge 0 \tag{19}$$

which is the classical Legendre condition, and

$$\int_{t_0}^{T} \eta^*_i(t)\,\eta^*_j(t)\,dt = \begin{cases} 0 \text{ if } i \ne j \\ \mu_i \text{ if } i = j \end{cases} \tag{20}$$

To simplify the subsequent analysis we assume that the trajectory is "strictly nonsingular," that is, $\xi^*_0(t)$ is strictly greater than zero. Then, without loss of generality, it may be assumed that $\xi^*_0(t) = 1$, for we now choose the normalizing factor $\alpha(t)$ of Eq. (4) to make this true. Thus we have the definition of $\alpha(t)$, leading to a somewhat simpler form of the equations to follow. Indeed, with $\xi^*_0(t) = 1$ it is straightforward to show that the envelope of points reachable in the boundary function space is, to second order, explicitly given by the paraboloid[4]

$$\delta\beta^*_0 = \frac{1}{2}\sum_{i=1}^{r} \frac{(\delta\beta^*_i)^2}{\mu_i} \tag{21}$$

---

[2]If the final time T were unspecified it would also be necessary that $\bar{\nu}'\left[\left(\frac{\partial\bar{\beta}}{\partial\bar{x}}\right)\bar{\mathrm{f}} + \left(\frac{\partial\beta}{\partial T}\right)\right] = 0.$

[3]Note that the second necessary condition would become the stronger Weierstrass condition if we pursued the analysis without the assumption that the magnitude of the impulse $\delta y(t)$ is small; that is, if we assumed that only the integral of the impulse were small. The Legendre condition is sufficient for our purposes, however, since we are considering here only weak (small) variations of the control function. With the simplified system model assumed, the first and second necessary conditions actually are sufficient conditions to establish a local optimum. A perturbation analysis of the more general system model is presented in Ref. 5 through 7, where it is shown that a condition analogous to the classical Jacobi condition must also be satisfied.

[4]Every point on this paraboloid represents one member of a "field" of extremal trajectories which exist in the neighborhood of the standard trajectory.

The $\{\mu_i\}$, which are the eigenvalues of the M matrix introduced in Eq. (15), are the radii of curvature of the envelope at the origin in the $(\delta\beta_0, \delta\beta_i)$ plane. It can be seen that the orthonormal matrix L rotates the coordinate axes in the boundary function space to coincide with the principal axes of reachable envelope.

IV. CONTROLLABILITY

The concepts of abnormality and uncontrollability (Ref. 8 and 9) are discussed in Ref. 1 and 6. It is pointed out that any optimal trajectory is always first order uncontrollable, which means that $\delta\beta_0^*$ cannot be driven to zero when given some arbitrary $(\delta\beta_0^*)^0$ if only the linear control terms of Eq. (10) are considered [recall that $\eta_0^*(t) = 0$]. Therefore, it is legitimate to denote the β_0^* direction as the "uncontrollable direction" in the boundary function space, where this direction is determined by the Lagrange multiplier vector

$$\bar{\nu}' = [\nu_0, \nu_1, \ldots \nu_r]$$

This vector $\bar{\nu}$ is normal to the reachable envelope at the origin. It is also true that any other $\delta\beta_i^*$ is first order uncontrollable if the corresponding $\mu_i = 0$ in Eq. (20), and this corresponds to the "abnormal" case in the classical calculus of variations (Ref. 8, page 210).[5] To eliminate difficulties we shall, henceforth, assume that the trajectory is normal for all $t_0 < T$. The trajectory always approaches abnormality as $t_0 \to T$, however; for from Eq. (15) we have

$$M \cong \bar{\eta}(T)\, \bar{\eta}'(T)\, [T - t_0] \qquad \text{for small } [T - t_0] \qquad (22)$$

which is an $(r + 1)$ by $(r + 1)$ matrix of rank 1.

The natural notion of second order controllability is introduced in Ref. 1, where an initial condition disturbance $\delta\bar{\beta}^0$ is said to be second order controllable if its negative image in the boundary function space lies within the reachable envelope. If this condition applies it follows that the origin $\delta\bar{\beta} = 0$ can be reached with the given $\delta\bar{\beta}^0$; that is, there is a $\Delta\bar{\beta}$ such that $\Delta\bar{\beta} + \delta\bar{\beta}^0 = 0$.

[5]The definition of abnormality presented here is slightly different from that presented in Ref. 8, where $\beta_0^* = \beta_1$ is also considered to be abnormal. This case corresponds to rotating the axes of the boundary function space (by the transformation L) through an angle of precisely $\pi/2$.

V. LEAST SQUARES CONTROL

It was pointed out in the previous part that all combinations of boundary conditions are not reachable by varying the control function; indeed, near the final time the trajectory becomes abnormal (first order uncontrollable) of order r. Given an initial condition error $\delta\bar{x}(t_0)$, which maps to a boundary function variation $\delta\bar{\beta}^0$, it is therefore reasonable to seek a "least squares" control scheme which seeks to minimize the magnitude of the final boundary function vector; that is, we minimize the performance index

$$p = \sum_{i=0}^{r} \delta\beta_i^2 = |\delta\bar{\beta}|^2 = |\delta\bar{\beta}^*|^2 \tag{23}$$

where $\delta\bar{\beta}$ is given by Eq. (10). Setting the first variation of p with respect to $\delta y(t)$ equal to zero, it follows that the control to be applied is given by

$$\delta y(t) = - \; (\delta\beta_0^*)^{-1} \sum_{i=1}^{r} \eta_i^*(t)\, \delta\beta_i^* \tag{24}$$

Multiplying Eq. (24) by $\eta_j^*(t)$, integrating, and solving for $\delta\beta_i^*$ we have

$$\delta\beta_i^* = \frac{(\delta\beta_i^*)^0 \, (\delta\beta_0^*)}{\delta\beta_0^* + \mu_i} \qquad i = 1, \ldots r \tag{25}$$

The $\delta\beta_i^*$ of Eq. (24) and (25) are the final values to be obtained, which can be found by dropping a perpendicular from $-(\delta\bar{\beta}^*)^0$ to the reachable envelope (Fig. 3). To show this, we multiply Eq. (24) by $[\delta y(t)\; \delta\beta_0^*]$ and integrate to obtain

$$2\Delta\beta_0^*\, \delta\beta_0^* + \sum_{i=1}^{r} \Delta\beta_i^*\, \delta\beta_i^* = \bar{t}'\, \delta\bar{\beta}^* = 0 \tag{26}$$

where t is the tangent to the reachable envelope at $\Delta\beta^*$ [from Eq. (21)]. Although this nonlinear control scheme is easily dealt with graphically, it is cumbersome analytically (a cubic equation must be solved). In the Appendix simple approximations of Eq. (24) and (25) are developed.

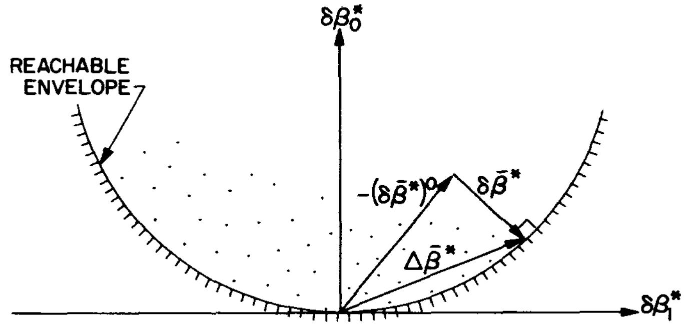

Fig. 3. Geometrical interpretation of least squares control

If all $\mu_i >> (\delta\beta_0^*)$, Eq. (25) yields

$$\delta\beta_i^* \cong [(\delta\beta_i^*)^0\, (\delta\beta_0^*)\, (\mu_i^{-1})]$$

and Eq. (24) becomes

$$\delta y(t) \cong - \sum_{i=1}^{r} \frac{(\delta\beta_i^*)^0\, \eta_i^*(t)}{\mu_i} \tag{27}$$

Thus we have very nearly a linear control law, which, from Eq. (16), (17), and (20), results in

$$\begin{aligned} \delta\beta_i^* &\cong 0 \qquad i = 1, \ldots, r \\ \delta\beta_0^* &\cong (\delta\beta_0^*)^0 \end{aligned} \tag{28}$$

As the initial time t_0 approaches the final time T, Eq. (22) shows that all influence functions except one approach zero. Taking this to be $\eta_1^*(t)$, we have

$$\lim_{t_0 \to T} \eta_i^*(t) = \begin{cases} |\bar{\eta}(T)| & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases} \tag{29}$$

Equation (15) shows that $\mu_i \to 0$ for $i = 0, 1, \ldots, r$ as $t_0 \to T$. Thus Eq. (24) and (25) become

$$\lim_{t_0 \to T} (\delta\beta_i^*) = (\delta\beta_i^*)^0 \qquad i = 0, 1, \ldots, r \tag{30}$$

$$\lim_{t_0 \to T} \delta y(t) = \left(\frac{\delta\beta_1^*}{\delta\beta_0^*} \right)^0 |\bar{\eta}(T)| \tag{31}$$

If $(\delta\beta_0^*)^o \to 0$ we have $\delta y(T) \to \infty$, which is the correct response. Note that the control at the final time T is stable, in the sense that its magnitude does not go to infinity for arbitrary disturbances $(\delta\beta_i^*)^o$, $i = 1, .. r$.

It could be observed that $\delta\bar{\beta}^* = 0$ also satisfies the least squares control criterion, and this result can be achieved if the initial condition is second order controllable. In this case the least squares control scheme does not yield the absolute minimum value of $|\delta\bar{\beta}^*|$, only a relative minimum. It does, however, yield a value of $|\delta\bar{\beta}^*|$ which is less than or equal to that obtained with any other optimal control method, that is, a smaller $|\delta\bar{\beta}^*|$ than would occur if we minimized some linear combination of the $\delta\beta_i^*$ while setting some other r linear combinations equal to zero. This conclusion, which follows from the above discussion, is verified graphically in Fig. 4 for the case of only one boundary function constraint ($r = 1$).

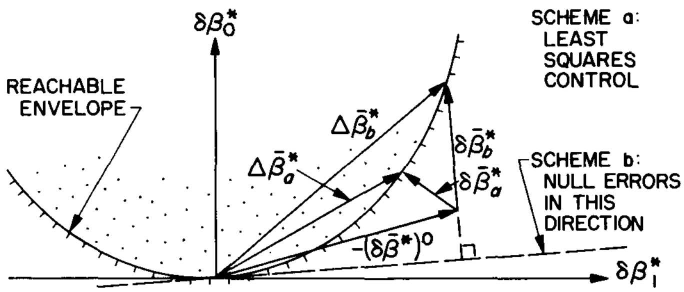

Fig. 4. Geometrical comparison of two control schemes

VI. ASCENT GUIDANCE INTO ORBIT

As an application of the least squares control technique we consider the task of guiding a vehicle into a near-Earth satellite orbit. In order to obtain closed form solutions we assume that the thrust acceleration magnitude is constant and that the gravitational acceleration can be approximated by a constant vector. The constant gravity assumption is a well known and reasonably accurate approximation for studies where the powered flight trajectory subtends a relatively small arc over the gravitating body. Choosing the x_4 axis along the gravity vector, the equations of motion become

$$\frac{d}{dt}(\bar{x}) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} a \cos\theta \\ a \sin\theta - g \\ x_1 \\ x_2 \end{bmatrix} = \bar{f}(\bar{x}, \theta, t) \qquad (32)$$

where

a is the given constant thrust acceleration; g is the constant gravitational acceleration; $\theta(t)$ is the steering angle to be determined.

We assume the standard steering program $\theta_s(t)$ has been chosen to minimize

$$\beta_0 = -x_1(T) \qquad (33)$$

subject to

$$\beta_1 = x_2(T) = 0 \qquad (34)$$

$$\beta_2 = g/v\,(x_4 - r) = 0 \qquad (35)$$

where

v and r are, respectively, the desired values of speed and altitude at injection into orbit. The factor g/v converts the position component x_4 to physical dimensions of speed, where this particular arbitrary factor was suggested by the energy relationship δ(specific energy) $= \delta\left(\frac{v^2}{2} + gr\right) = v\left(\delta v + \frac{g}{v}\delta r\right)$. The performance index to be minimized on the standard trajectory is given by

$$p = \nu_0\beta_0 + \nu_1\beta_1 + \nu_2\beta_2 \qquad (36)$$

where ν_i is the Lagrange multiplier and $\sum_{i=0}^{2} \nu_i^2 = 1$.

To further simplify the forthcoming calculations, let us suppose that the x_1 axis coincides with the direction of the thrust vector on the standard trajectory at the final time T. Thus, $\theta_s(T) = 0$; and it follows that $\nu_1 = 0$. We define the control variable to be [Eq. (4)][6]

$$y = \theta \left[\frac{\nu_0 a}{\cos \theta_s(t)} \right]^{1/2} \tag{37}$$

It is easily verified that the state transition matrix corresponding to Eq. (32) is

$$U(T,t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \tau & 0 & 1 & 0 \\ 0 & \tau & 0 & 1 \end{bmatrix} \tag{38}$$

where $\tau = (T - t)$. The influence function vector is

$$\bar{\eta}(t) = \begin{bmatrix} \eta_0(t) \\ \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \left[\frac{\partial \bar{\beta}}{\partial \bar{x}} \right] [U(T,t)] \left[\frac{\partial \bar{f}}{\partial \theta} \right] \left[\frac{d\theta}{dy} \right]$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{g}{v} \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ -\tau \sin\theta \\ \tau\cos\theta \end{bmatrix} \left(\frac{\cos \theta_s(t)}{\nu_0 a} \right)^{1/2} (a)$$

$$= \left(\frac{a \cos_s(t)}{\nu_0} \right)^{1/2} \begin{bmatrix} \sin\theta \\ \cos\theta \\ \dfrac{g\tau}{v} \cos\theta \end{bmatrix} \tag{39}$$

If the performance index [Eq. (36)] is to be minimized it follows that on the standard trajectory we must have

$$0 = \eta_0^*(t) = \nu_0 \eta_0(t) + \nu_2 \eta_2(t)$$

$$= \left(\frac{a \cos \theta_s(t)}{\nu_0} \right)^{1/2} \left(\nu_0 \sin\theta + \nu_2 \frac{g\tau}{v} \cos\theta \right) \tag{40}$$

[6]The radicand is guaranteed to be greater than zero by the Legendre condition.

and hence

$$\tan \theta_s(\tau) = -\left(\frac{\nu_2}{\nu_0} \right) \left(\frac{g\tau}{v} \right) \tag{41}$$

Equation (19) becomes

$$\xi_0^*(t) = \left(\frac{\partial \eta_0^*(t)}{\partial y} \right) = \left(\frac{\partial \eta_0^*(t)}{\partial \theta} \right) \left(\frac{d\theta}{dy} \right) \tag{42}$$

$$= \left[\frac{\cos \theta_s(t)}{\nu_0} \right] \left[\nu_0 \cos \theta_s(t) - \left(\frac{\nu_2 g\tau}{v} \right) \sin \theta_s(t) \right] = 1$$

The transformation $L(t_0)$ of Eq. (14) is (recall that $\nu_0^2 + \nu_2^2 = 1$)

$$L(t_0) = \begin{bmatrix} \nu_0 & 0 & \nu_2 \\ -\nu_2 \sin\psi(t_0) & \cos\psi(t_0) & \nu_0 \sin\psi(t_0) \\ -\nu_2 \cos\psi(t_0) & -\sin\psi(t_0) & \nu_0\cos\psi(t_0) \end{bmatrix} \tag{43}$$

where

$$\tan 2\psi(t_0) = \left\{ \frac{2\nu_2 [1 - \cos\theta_s(t_0)]}{\ln \tan\left[\dfrac{\pi}{4} + \dfrac{\theta_s(t_0)}{2} \right] - [\sin\theta_s(t_0)]\,[1 + \nu_2^2]} \right\} \tag{44}$$

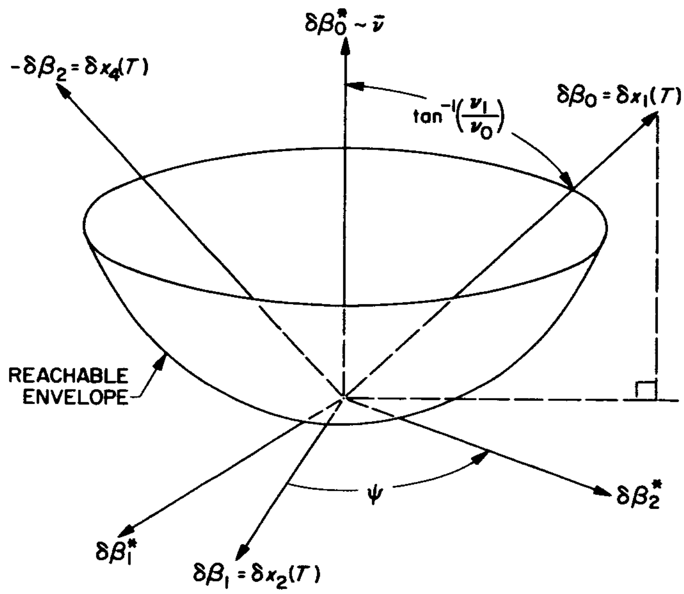

Fig. 5. Boundary function space for the ascent guidance problem

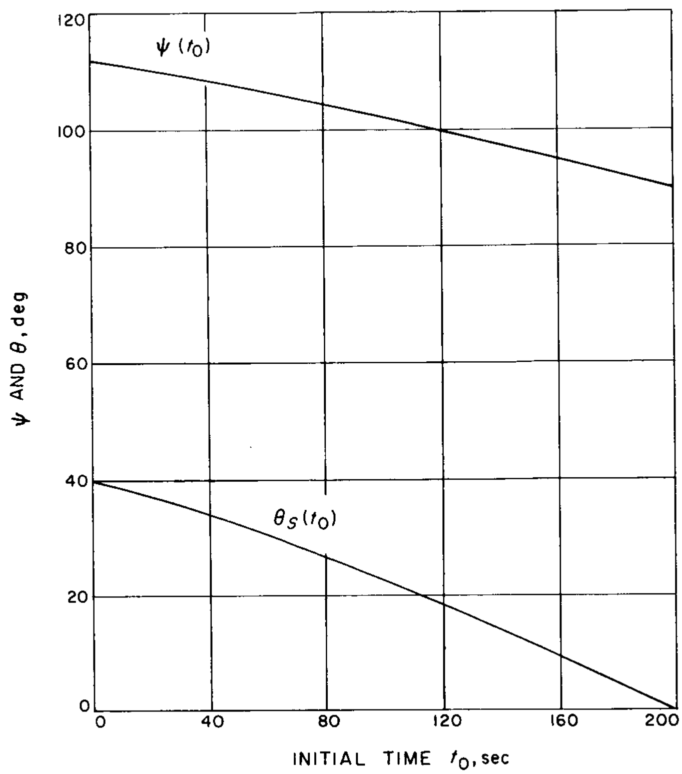

Fig. 6. The transformation and standard steering angles

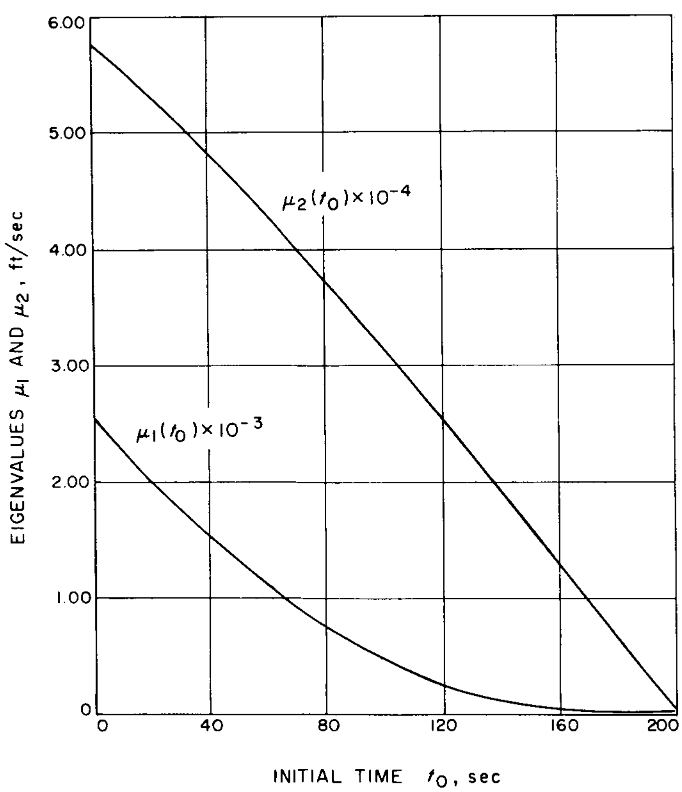

Fig. 7. Eigenvalues of the *M* (t_0) matrix

$$\delta\bar{\beta}^* = L(t_0)\,\delta\bar{\beta} = L(t_0)\left[\frac{\partial\bar{\beta}}{\partial\bar{x}}\right]\delta\bar{x}(T) \tag{45}$$

$$\bar{\eta}^*(t) = L(t_0)\bar{\eta}(t) \qquad t \geq t_0 \tag{46}$$

After some calculation, it can be verified that Eq. (20) holds. Figure 5 presents a picture of the boundary function space.

To demonstrate the effect of following the simplified version of the least squares control technique (see Appendix) a standard trajectory was constructed with $\nu_2 = -0.9559$, $g = 32.0$ ft/sec², a $= 96.0$ ft/sec², $v = 25{,}000$ ft/sec, and $T = 200$ sec. Various initial condition errors $\delta\bar{x}(t_0)$ were chosen for values of t_0 ranging from 0 to 200 sec; the above described calculations were carried out; the resultant $\delta\bar{\beta}^*$ was obtained; and $\delta\bar{\beta} = L'(t_0)\delta\bar{\beta}^*$ was computed. The auxiliary quantities $\theta_s(t)$, $\psi(t_0)$, $\mu_1(t_0)$, and $\mu_2(t_0)$ are shown in Fig. 6 and 7.

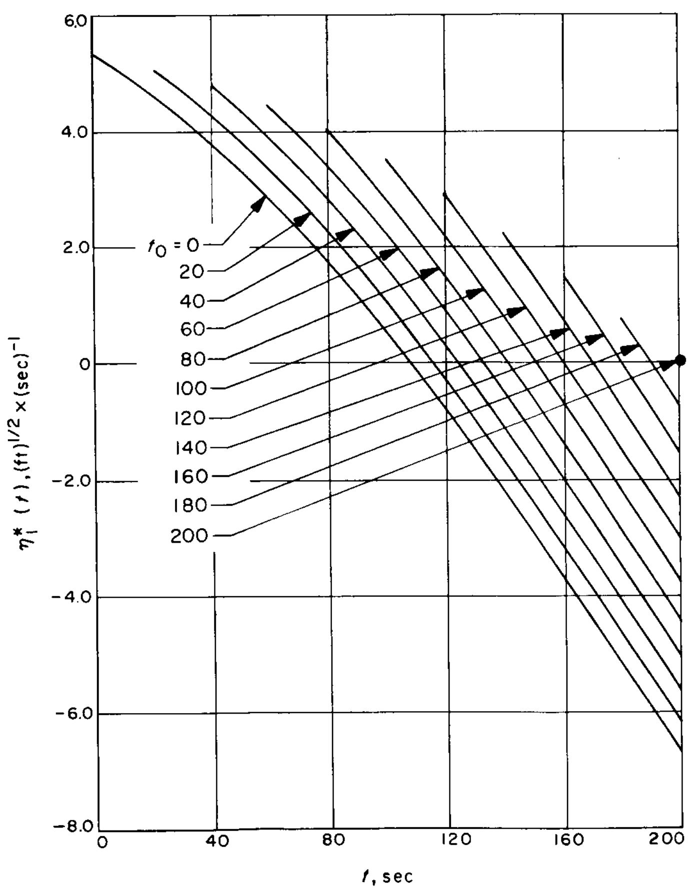

Fig. 8. Impulse response function η_1^* (*t*) for various initial times t_0

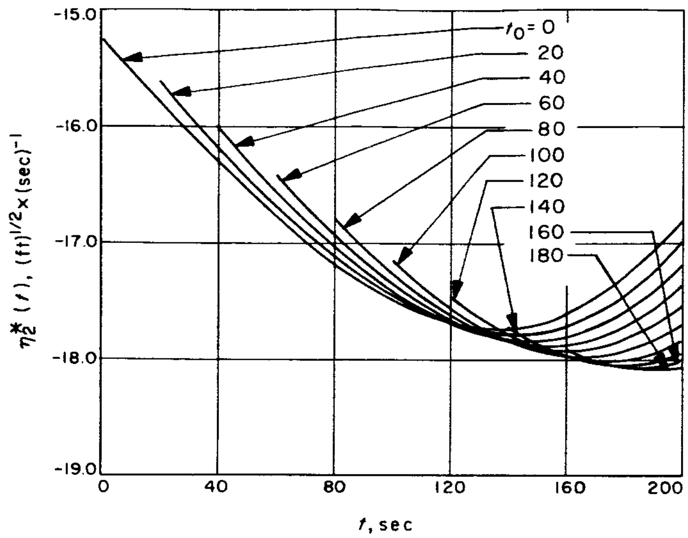

Fig. 9. Impulse response function $\eta_2^*(t)$ for various initial times t_0

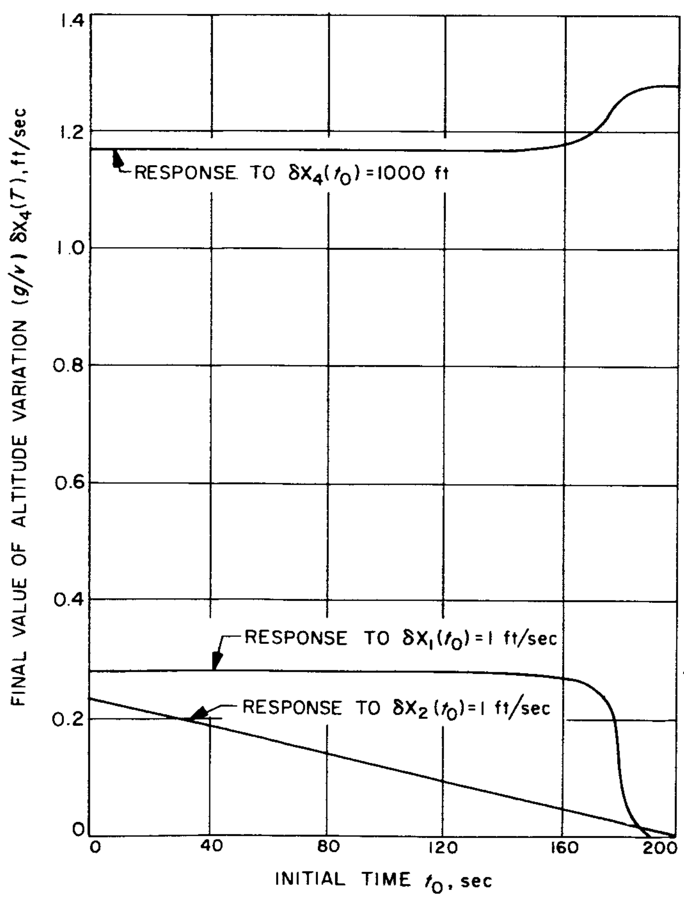

Fig. 10. Final values of altitude variation resulting from various initial condition disturbances

The influence functions $\eta_1^*(t)$ and $\eta_2^*(t)$ are plotted vs. t for various t_0 in Fig. 8 and 9. In Fig. 10 and 11 we have the $\delta x_4(T)$ and $\delta x_1(T)$ resulting from applying the least squares control technique to correct selected values of initial condition variations for various initial condition times t_0. The corresponding $\delta x_2(T)$ is not shown because it is negligibly small when compared to $\delta x_1(T)$. On Fig. 11 is superimposed the value of $\delta x_1(T)$ which is obtained

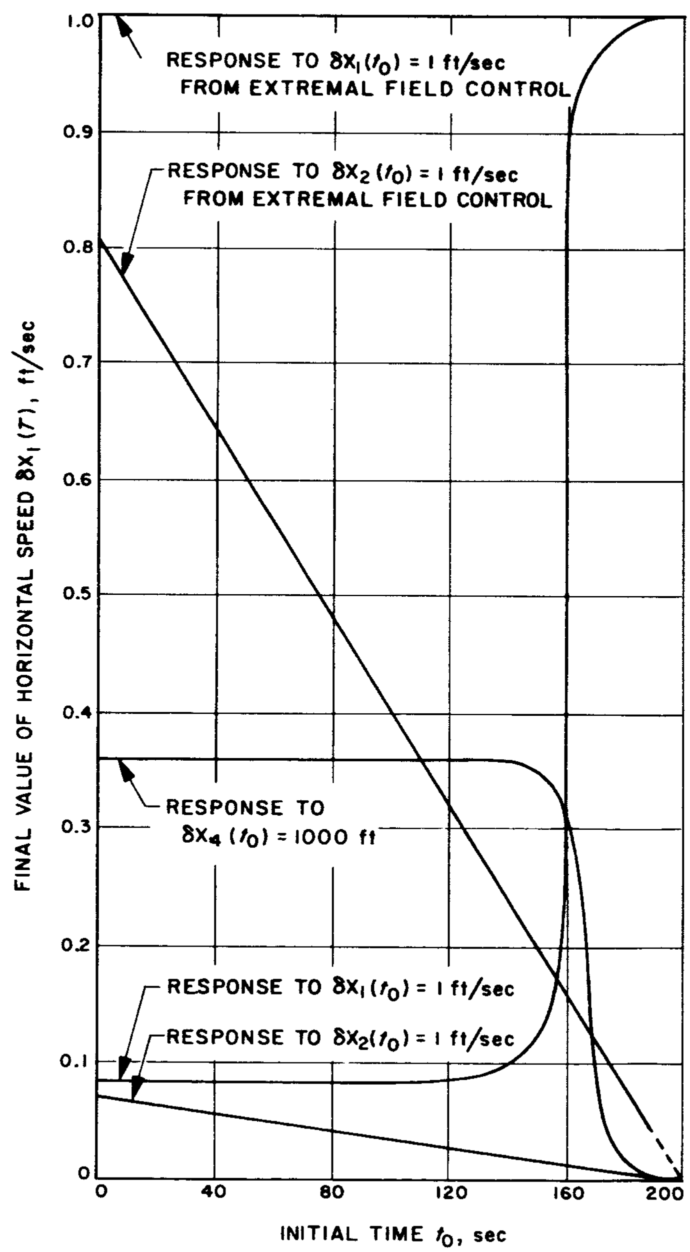

Fig. 11. Final values of horizontal speed variation resulting from various initial condition disturbances

from applying the extremal field control scheme described in Ref. 2 and 3,[7] where $\delta x_2(T)$ and $\delta x_4(T)$ are held equal to zero. (The response to $\delta x_4(t_0) = 1000$ ft is not shown because it is off the graph.) Note that the sum of the squares of the three final state variations for the least squares control scheme is always less than the $[\delta x_1(T)]^2$ resulting from extremal field control, as expected.

[7]These data were obtained from an unpublished work by Dr. Byron D. Tapley of the University of Texas.

VII. THE MINIMUM TIME TRAJECTORY

The analysis thus far has considered only the case of fixed final time, but the minimum time problem also fits within the framework of this analysis. In preparation for the analysis presented in Part VIII, this well known conclusion is discussed here.

Suppose we are to achieve the conditions $\beta_i[\bar{x}(T), T] = 0$ in minimum time, for $i = 0, 1, \ldots r$, where β_i is as described in Section II. We assume the existence of a standard trajectory with final time T_s, and for any neighboring perturbed trajectory we have, to first order,

$$d\bar{\beta} = \delta\bar{\beta}(T_s) + (\dot{\bar{\beta}}_s)dT \tag{47}$$

where

$$d\bar{\beta} = \bar{\beta}[\bar{x}(T), T] - \bar{\beta}[\bar{x}_s(T_s), T_s]$$

$$dT = T - T_s$$

$$\dot{\bar{\beta}}_s = \left[\left(\frac{\partial\bar{\beta}}{\partial\bar{x}}\right)(\dot{\bar{x}}_s) + \left(\frac{\partial\bar{\beta}}{\partial T}\right)\right]_{T_s}$$

$$= \left[\left(\frac{\partial\bar{\beta}}{\partial\bar{x}}\right)(\bar{f}) + \left(\frac{\partial\bar{\beta}}{\partial T}\right)\right]_{T_s}$$

Let us choose the $(r + 1)$ by $(r + 1)$ orthonormal matrix K, with first row equal to $(\dot{\bar{\beta}}_s')(|\dot{\bar{\beta}}_s|^{-1})$, and the other rows orthogonal to this vector and normalized but otherwise arbitrary. Premultiplying Eq. (47) by K we have

$$d\hat{\bar{\beta}} = Kd\bar{\beta} = \begin{vmatrix} \delta\hat{\beta}_0(T_s) \\ \delta\hat{\beta}_1(T_s) \\ \cdot \\ \cdot \\ \delta\hat{\beta}_r(T_s) \end{vmatrix} + \begin{vmatrix} |\dot{\bar{\beta}}_s(T_s)| \\ 0 \\ \cdot \\ \cdot \\ 0 \end{vmatrix} dT \tag{48}$$

Thus

$$\begin{aligned} dT &= [d\hat{\beta}_0 - \delta\hat{\beta}_0(T_s)]\,|\dot{\bar{\beta}}_s(T_s)|^{-1} \\ d\hat{\beta}_i &= \delta\hat{\beta}_i(T_s) \qquad i = 1, \ldots r \end{aligned} \tag{49}$$

Equation (49) implies that dT can be made nonzero if there exists a control variation which, to first order, yields $\delta\hat{\beta}_0(T_s) \neq 0$ and $\delta\hat{\beta}_i(T_s) = 0$ for $i = 1, \ldots r$. Thus we conclude that satisfying $\beta_i[\bar{x}(T), T] = 0$ for $i = 0, 1, \ldots r$ in minimum time is equivalent to minimizing $\hat{\beta}_0[\bar{x}(T), T]$ in the fixed time T_s, subject to $\hat{\beta}_i[\bar{x}_s(T_s), T_s] = 0$ for $i = 1, \ldots r$. Thus we proceed as in the fixed time case, where

$$\delta\bar{\beta}^* = [\hat{L}(t_0)]\,\delta\hat{\bar{\beta}} = [\hat{L}(t_0)\,K]\,\delta\bar{\beta} \triangleq L\delta\bar{\beta}$$

VIII. MINIMIZING VELOCITY TO BE GAINED

The well known velocity-to-be-gained guidance scheme can be developed by supposing that there is a moving target point to be intercepted at some future time $t_f > T_s$, where T_s the time of thrust termination on the standard trajectory. Letting the position components be denoted by $\bar{r}' = [x_3, x_4]$ and the velocity coordinates by $\bar{v}' = [x_1, x_2]$, the guidance task at thrust termination is to null the predicted target error; that is, set $\delta\bar{r}(t_f) = \bar{r}(t_f) - \bar{r}(t_f)_{\text{target}} = 0$. We imagine applying a corrective velocitv impulse $\Delta\bar{v}$ at T_s to obtain

$$\delta\bar{r}(t_f) = 0 = \left[\frac{\partial\bar{r}(t_f)}{\partial\bar{r}(T_s)}\right]\delta\bar{r}(T_s) + \left[\frac{\partial\bar{r}(t_f)}{\partial\bar{v}(T_s)}\right][\delta\bar{v}(T_s) + \Delta\bar{v}] \tag{50}$$

The term $\Delta\bar{v}$ is called the "velocity to be gained" for the given $\delta\bar{r}$ and $\delta\bar{v}$, and from Eq. (50) is found to be

$$-\Delta\bar{v} = \delta\bar{v}(T_s) + \left[\frac{\partial\bar{r}(t_f)}{\partial\bar{v}(T_s)}\right]^{-1}\left[\frac{\partial\bar{r}(t_f)}{\partial\bar{r}(T_s)}\right]\delta\bar{r}(T_s) \tag{51}$$

The velocity to be gained is physically realized by varying the thrust termination time by a small amount ΔT, thus

$$\Delta\bar{v}' = a(T_s)[\cos\theta(T_s), \sin\theta(T_s)]\,\Delta T \tag{52}$$

where

$$\tan\theta(T_s) = \left[\frac{\Delta v_2}{\Delta v_1}\right] \tag{53}$$

$$\Delta T = |\Delta\bar{v}|\,[a(T_s)]^{-1} \tag{54}$$

and Δv_1, Δv_2 are obtained from Eq. (51). In practical applications Eq. (53) and (54) are employed to solve for θ and ΔT near the final time T_s, with steering during the early phases of flight constructed (usually pragmatically) in such a fashion as to minimize the predicted magnitude $|\Delta\bar{v}|$.

Let us show that the velocity-to-be-gained guidance scheme can be obtained by applying the least squares control technique discussed above. We consider the problem of attaining in minimum time an $\bar{x}(T_s)$ which will cause $\delta\bar{r}(t_f) = 0$. Applying the results of the previous section, with $\Delta\beta_0 = \Delta v_1$ and $\Delta\beta_1 = \Delta v_2$, we have from Eq. (51),

$$\left[\frac{\partial\bar{\beta}}{\partial\bar{x}(T_s)}\right] = -\left\{I \;\vdots\; \left[\frac{\partial\bar{r}(t_f)}{\partial\bar{v}(T_s)}\right]^{-1}\left[\frac{\partial\bar{r}(t_f)}{\partial\bar{r}(T_s)}\right]\right\} \tag{55}$$

where I is the 2 by 2 identity matrix and ($\vdots$) indicates a matrix partitioning. Employing least squares control, we find from Eq. (31) that the final value of the control variation is given by

$$\delta y(T_s) = \left[\frac{\delta\theta(T_s)}{\alpha(T_s)}\right] = \left[\frac{\Delta v_2}{\Delta v_1}\right]|\bar{\eta}(T_s)| \tag{56}$$

where

$$\begin{aligned}|\bar{\eta}(T_s)|^2 &= [\alpha(T_s)\,a(T_s)]^2\,[\sin^2\theta_s(T_s) + \cos^2\theta_s(T_s)] \\ &= [\alpha(T_s)\,a(T_s)]^2\end{aligned} \tag{57}$$

Without loss of generality we choose the x_1 coordinate axis to be parallel to the thrust vector at T_s on the standard trajectory. It follows that $\theta_s(T_s) = 0$, $\nu_0 = 1$, $\nu_2 = 0$, and $\alpha^2(T_s) = \left[\frac{1}{a(T_s)}\right]$. From Eq. (56) and (57) we now have

$$\delta\theta(T_s) = \alpha^2(T_s)\,a(T_s)\left(\frac{\Delta v_2}{\Delta v_1}\right) = \left(\frac{\Delta v_2}{\Delta v_1}\right) \tag{58}$$

Equation (58) is the expression for θ which would be obtained from expanding to first order the left hand side of Eq. (53). If we agree to adjust ΔT by Eq. (54) it follows that, to first order, the velocity-to-be-gained steering technique can be thought of as a special case of least squares control.

IX. CONCLUSION

A final value control scheme has been presented which minimizes the magnitude of the variation in the boundary function vector. It was shown that the control scheme is stable at the final time, and a correspondence between this approach and the well known velocity-to-be-gained steering scheme has been established. Although only a simplified set of perturbation equations have been analyzed, the extension to include the second-order state variation terms, and the state variation/control variation terms, is not theoretically difficult. This mathematical model is discussed in Ref. 1, 5, 6 and 7.

The control problem treated here is related to the task of minimizing a quadratic function of the final state variables for a linear, completely controllable system (Ref. 9). Because of the optimality of the standard trajectory, however, the perturbation equations we consider always yield a dynamic system which is guaranteed first order uncontrollable. This particular property is of central importance, because, if the system were completely controllable, the best least squares control would be attained by simply nulling the end condition variations. A control law which is linear in the state variations would accomplish this purpose. Unlike this well-known result, the approach taken here yields a nonlinear control law, which nulls the final state variations "as closely as possible." It is interesting that Eq. (27) and (28) show that, if the eigenvalues μ_i are sufficiently large, the least squares control approximately nulls the r controllable boundary conditions and leaves the uncontrollable component unchanged. This is the solution one would intuitively expect.

It should be noted that the numerical results obtained depend upon the metric chosen for the boundary function space, that is, the weighting factors assigned to the individual boundary function variations. This choice might be made empirically or be based upon physical reasoning. In any case the sum of the squares of the boundary function variations would be less than or equal to that obtained for any other control scheme having the same metric.

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APPENDIX

A derivation of simplified expressions for the least squares control variation, and the resulting $\delta\beta_i^*$ [Eq. (24), (25)] is presented in this Appendix.

We consider first the case of one constraint ($r = 1$) and, to simplify notation, define $\delta\beta_1^* = x$ and $\delta\beta_0^* = y$. Given

$$\bar{a}' = [a_1, a_2] = \left[(-\delta\beta_1^*)^0, (-\delta\beta_0^*)^0\right]$$

we seek the vector

$$\bar{b}' = [b_1, b_2] = \left[\delta\beta_1^*, \delta\beta_0^*\right]$$

which is the perpendicular dropped from the tip of $\bar{a}$ to the reachable envelope $y = \left(\frac{x^2}{2\mu}\right)$ (Fig. 3). If

$$\bar{c}' = [c_1, c_2] = \left[\Delta\beta_1^*, \Delta\beta_0^*\right]$$

we have

(1) $\bar{c} = \bar{a} + \bar{b}$

(2) $\bar{c}$ touches the parabola $y = \dfrac{x^2}{2\mu}$

(3) $\bar{b}$ is orthogonal to the parabola at the contact point

Thus, $\bar{b}$ must be of the form

$$\bar{b}' = k\left[\frac{c_1}{\mu}, -1\right] \tag{A-1}$$

Where k is a proportionality factor, and we have the following two simultaneous equations for c_1 and k:

$$a_1 + \frac{kc_1}{\mu} = c_1 \tag{A-2}$$

$$a_2 - k = c_2 = \frac{c_1^2}{2\mu} \tag{A-3}$$

Eliminating k, we obtain the cubic equation

$$c_1^3 + 2\mu(\mu - a_2)c_1 - 2\mu^2 a_1 = 0 \tag{A-4}$$

Equation (25) becomes

$$b_1 = \left(\frac{-a_1 b_2}{c_2 - a_2 + \mu}\right) = -\left(\frac{c_1}{\mu}\right)(b_2) \tag{A-5}$$

If

$$d = \mu^3\left\{a_1^2\mu + \left[\frac{2}{3}(\mu - a_2)\right]^3\right\} \tag{A-6}$$

then

$d > 0$ implies that Eq. (A-4) has one real and two imaginary roots.

$d \leq 0$ implies that Eq. (A-4) has three real roots

The case $d \leq 0$ can only occur for relatively small values of μ, which occurs when the initial time t_0 approaches the final time T. In this instance we arbitrarily choose the solution of Eq. (A-4) which yields the smallest value of $|c_1|$, thereby assuring that (c_1/μ) always goes to $-(a_1/a_2)$ at the final time.

To avoid solving the cubic equation, we seek an approximate expression for Eq. (A-5). Writing Eq. (A-4) in the form

$$-\left(\frac{c_1}{\mu}\right) = \left(\frac{-a_1}{\mu - a_2}\right)\left[1 - \left(\frac{c_1^3}{2\mu^2 a_1}\right)\right] \tag{A-7}$$

we have

$$-\left(\frac{c_1}{\mu}\right) \cong \left(\frac{-a_1}{\mu - a_2}\right) = \frac{(\delta\beta_1^*)^0}{\mu + (\delta\beta_0^*)^0} \tag{A-8}$$

if

$$\left|\frac{\mu a_1^2}{2(\mu - a_2)^3}\right| \leq \varepsilon \tag{A-9}$$

where ε is a small number. The test Eq. (A-9) can only be violated when μ becomes the same order of magnitude as a_2, that is, when μ is small. In that instance we further approximate the correct result by arbitrarily setting $\mu = 0$.

For the multiconstraint case ($r > 1$) we employ the same approximation technique for each $\delta\beta_i^*$, $i = 1, \ldots r$, resulting in the control variation [Eq. (24)]

$$\delta y(t) = -\sum_{i=1}^{r} \frac{\eta_i^*(t)\,(\delta\beta_i^*)^0}{(\delta\beta_0^*)^0 + \mu_i} \tag{A-10}$$

where a test of the type Eq. (A-9) must be applied to each μ_i. From Eq. (A-8) it follows that the approximate expression for $\delta\beta_i^*$ is of the form of Eq. (25), except that $(\delta\beta_0^*)$ is replaced with $(\delta\beta_0^*)^0$. From Eq. (16) and (A-10) it follows that the approximate expression for $(\delta\beta_0^*)$ is given by

$$\delta\beta_0^* = (\delta\beta_0^*)^0 + \sum_{i=1}^{r} \left[\frac{(\delta\beta_i^*)^0}{(\delta\beta_0^*)^0}\right]^2 \mu_i \tag{A-11}$$

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